The two circles of any of these pairs are in general position; the four circles tangent to them are those belonging to the other two pairs. Hence any one of the six circles is touched by four of the remaining circles, namely, by all except the one belonging to the same pair.

Pass a circle through any three of the four points of intersection, say P_2, P_3, P_4. This circle determines a unique sphere, passing through it and orthogonal so S, which may be denoted by S_1. This sphere, it is easily shown, passes through the circles C_{23}, C_{24}, C_{34}. In all there are four spheres, S_1, S_2, S_3, S_4 of this kind, each containing three of the six circles.

Let us consider the different ways in which a triple of circles can be selected.

1°. Two circles of the same pair and any other circle, for example, C_{12}, C_{34}, C_{13}; two of the circles are then in general position and the third touches both.

2°. One from each of the three pairs, but in such a way that the three circles have no common point, for example, C_{12}, C_{13}, C_{23}; the circles are then co-spherical (lying on one of the spheres S_1, S_2, S_3, S_4) and are mutually tangent.

3°. Three having a point in common, for example, C_{12}, C_{13}, C_{14}; the circles are then tangent to one another at the common point.

The numbers of triples for the three types are respectively 12, 4 and 4. If we consider in each case the residual triple, for 1° it is of the same type; while for 2° it is of type 3° and *vice versa*.

The problem considered belongs essentially to the geometry based upon the group of all conformal transformations in space, namely, the group generated by inversions or transformations by reciprocal radii vectors. From the point of view of this geometry the infinite region of space is regarded as a point, instead of a plane, as in projective geometry; a straight line is thought of as a special circle, namely, one which passes through the infinite point; and similarly a plane is a special sphere.

In the preceding discussion of the problem, it was assumed that the two given circles were in general position. The enumeration of all the special cases is not difficult and makes an interesting exercise, It will suffice to state that, if the two circles intersect at a single point (including, for instance, the case where the circles become two straight lines in general position), there is no proper solution (all the four circles shrinking up into the point of intersection). On the other hand, if the circles have two real or imaginary points in common, *i. e.*, if they are co-spherical, the problem becomes indeterminate, for then there are an infinite number of tangent circles just as in the case of two circles in plane geometry.

COLUMBIA UNIVERSITY, *May 7, 1903.*

ON THE DEFINITION OF AN INFINITE NUMBER.

By DR. G. A. MILLER.

Mathematics is replete with surprises. Three of these are especially noteworthy on account of the fundamental principles involved, namely, (1) There are perfectly consistent geometries in which the sum of the angles of a plane triangle is not equal to two right angles. (2) There are algebras in which the commutative law of multiplication does not hold. (3) There are aggregates such that a part is equivalent to the whole.

The first of these has been frequently noted in this journal. In regard to the second, Poincaré recently said*: "Hamilton's quaternions give us an example of an operation which presents an almost perfect analogy with multiplication, which may be called multiplication, and yet it is not commutative; *i. e.*, the product is changed when the order of the factors is changed. This presents a revolution in arithmetic which is entirely similar to the one which Lobatchevski effected in Geometry."

The object of the present note is to give some details in regard to an infinite aggregate. The method employed to compare two aggregates is one of the most rudimentary ones. In fact, it is essentially the same as that employed in the earliest stages of civilization when objects are counted by means of pebbles. In this way a (1, 1) correspondence is established between a given pile of pebbles and a certain aggregate of objects.

Similarly, two aggregates or totalities are said to be *equivalent* when it is possible to establish a (1, 1) correspondence between the units of the aggregates. That is, when the units of the aggregates can be so associated that to each unit of one of the aggregates there corresponds one and only one unit of the other and *vice versa*. For instance, every even integer is of the form $2n$, and every odd integer is of the form $2n+1$. For any integral value of n we obtain one and only one even number, and also one and only one odd number. By associating these two numbers for every value of n we obtain a (1, 1) correspondence between the odd and the even numbers. Hence we say that the totality of even numbers is equivalent to the totality of odd numbers.

A more interesting illustration may be given by means of the equation $y=1/(x+1)$. Suppose we assign to x any positive rational value. The corresponding value of y will always be a positive rational number which does not exceed unity. Whenever we give x two distinct values, y will assume two distinct values in the given intervals and *vice versa*. Hence it follows that *the rational positive numbers which do not exceed unity are equivalent to all the rational positive numbers.*

By employing the equation $2y=1/(x+1)$, it is easy to see that the totality of the positive rational numbers which do not exceed one-half is equivalent to

*Poincare, *Bulletin des Sciences Mathematiques*, Vol. 26, 1902, page 250.

the totality of all the positive rational numbers. More generally, the equation $ay=1/(x+1)$, a being a positive rational number, indicates that the totality of the positive rational numbers which do not exceed an arbitrary finite limit is equivalent to the totality of all the positive rational numbers.

The above examples furnish some of the most important instances of multitudes in which a part is equivalent to the whole. It should be observed that this equivalence is proved by one of the most elementary methods of comparison—a method which every one employs with perfect confidence.

With respect to any aggregate only two cases are thinkable; either a part is equivalent to the whole or there is no such part. In the former case the aggregate is said to be infinite, in the latter case it is said to be finite. This definition of an infinite number of things is due to Dedekind and is of fundamental importance.* That infinite numbers have this property was noted by others, but Dedekind was the first to use it as a definition.

In employing this definition any aggregate (M_1) is to be considered a part of another aggregate (M_2), provided each of the units of M_1 is contained in M_2, while there is at least one unit in M_2 which is not contained in M_1. For instance, all the even integers are a part of the total number of integers and all the integers are a part of all the rational numbers.

It is now easy to prove that the number of integers is infinite, for the equation $y=2x$ shows that there is just one even integer (y) which corresponds to any integer (x), and that there is just one integer x which corresponds to any even number y. That is, there is a (1, 1) correspondence between the total number of integers and the even integers. As the latter constitute a part of the former it follows that a part of the totality of integers is equivalent to the whole number of integers, that is, the number of integers is infinite.†

By the same method it may be proved that time, according to the common conception, is infinite; for there is a (1, 1) correspondence between the total number of hours and the total number of half hours. If the half hours were denoted by the natural numbers the hours would correspond to the even numbers. Similarly, it may be observed that space (defined analytically by coördinates) is infinite according to the given definition.

If it is possible to establish a (1, 1) correspondence between two infinite totalities they are said to be of the same power. The totality (M_0) formed by the natural numbers is especially important. Any totality which has the same power as M_0 is said to be numerable (countable). It is an extremely interesting fact that all algebraic numbers are countable. In particular, the natural numbers are equivalent to all the rational numbers. These facts constitute some of the elements of the important subject known as the theory of aggregates (ensemble, Mengenlehre) to which the reader may be referred for further developments.

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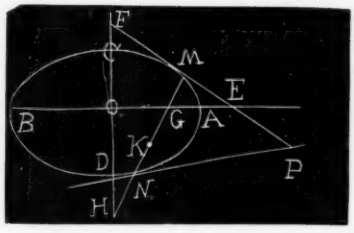
*Dedekind, *Essays on Number*, translated by Beman, 1901, page 63.

†This result can be proved in an infinite number of ways by using the equation $y=ax$, a being any integer. It may also be proved by the equations $y=x^a$, $y=x+a$, etc.

CERTAIN LOCI RELATED TO A CONIC.

By G. B. M. ZERR.

Let M be any point on an ellipse, axes $2a$, $2b$, eccentricity e. O the center of the ellipse, MP the tangent, MN the normal, and MQ the diameter, respectively, at the point M. Let E, F be the points where the tangent MP intersects the axes; G, H the points where the normal intersects the axes. P the pole of MN; K the center of curvature. Let (u, v) be the coördinates of M. Then $(-u, -v)$; $(a^2/u, 0)$; $(0, b^2/v)$; $(e^2u, 0)$; $(0, -a^2e^2v/b^2)$; $(e^2u^3/a^2, -a^2e^2v^3/b^4)$; $[a^2/(e^2u), -b^4/(a^2e^2v)]$; $[u(1-b^2p), v(1-a^2p)]$, where $p=2\left(\frac{a^4v^2+b^4u^2}{a^6v^2+b^6u^2}\right)$, are the coördinates of Q, E, F, G, H, K, P, and N, respectively.



The equation of the perpendicular from O on MP is $y=a^2vx/b^2u$.....(1).

The perpendicular from K on OG is $x=e^2u^2/a^2$.....(2).

The perpendicular from G on OM is $y+\frac{u}{v}(x-e^2u)=0$.....(3).

The perpendicular from K on OH is $y+a^2e^2v^3/b^4=0$.....(4).

The perpendicular from H on OM is $y+a^2e^2v/b^2+ux/v=0$.....(5).

The perpendicular from E on MP is $y-\frac{a^2v}{b^2u}\left(x-\frac{a^2}{u}\right)$.....(6).

The perpendicular from K on MN is $y+\frac{a^2e^2v^3}{b^4}+\frac{b^2u}{a^2v}\left(x-\frac{e^2u^3}{a^2}\right)=0$.....(7)

The perpendicular from F on MP is $y=b^2/v+a^2vx/b^2u$.....(8).

The perpendicular from K on OM is $y+\frac{a^2e^2v^3}{b^4}+\frac{u}{v}\left(x-\frac{e^2u^3}{a^2}\right)=0$..... (9).

The perpendicular from H on NM is $y+\frac{a^2e^2v}{b^2}+\frac{b^2ux}{a^2v}=0$.....(10).

The perpendicular from G on MN is $y+\frac{b^2u}{a^2v}(x-e^2u)=0$.....(11).

The perpendicular from M on OG is $x=u$.....(12).

The perpendicular from M on OH is $y=v$.....(13).

The perpendicular from M on OM is $y-v+\frac{u}{v}(x-u)=0$.....(14).

The line PG is $y(a^4v-a^2e^4u^2v)+b^4ux-b^4e^2u^2=0$.....(15).

The line PH is $y+\frac{a^2e^2v}{b^2}=\frac{(a^4e^4uv^2-b^6u)x}{a^4b^2v}$.....(16).

The line QN is $a^4vy+b^4ux+a^4v^2+b^4u^2=0$.....(17).

The line OK is $y+\frac{a^4v^3x}{b^4u^3}=0$.....(18).

The line through the projections of Q on OG and OH is $uy+vx+uv=0$....(19).

Then we have the following eleven sets of three concurrent lines:

1. (1), (2), (3) meet in $x=e^2u^3/a^2$, $y=u^2v/b^2$.

$\therefore u=\sqrt[3]{\frac{a^2x}{e^2}}$, $v=b^2y\sqrt[3]{\frac{e^4}{a^4x^2}}$. Substituting these values of u, v in $b^2u^2+a^2v^2=a^2b^2$, we get for the locus of this point $(a^2x^2+b^2e^4y^2)^3=a^8e^4x^4$.

2. (1), (4), (5) meet in $x=-\frac{e^2uv^2}{b^2}$, $y=-\frac{a^2e^2v^3}{b^4}$.

$\therefore u=-\frac{ax}{e}\sqrt[3]{\frac{ae}{b^2y^2}}$, $v=-b\sqrt[3]{\frac{by}{a^2e^2}}$.

$\therefore$ The locus of this point is $(a^2x^2+b^2y^2)^3=a^4b^4e^4y^4$.

3. (9), (10), (12) meet in $x=u$, $y=-\frac{a^4e^2+b^2u^2-a^2e^2u^2}{a^2v}$.

$\therefore u=x$, $v=-\frac{a^2e^2+b^2x^2-a^2e^2x^2}{a^2y}$.

The locus of this point is $a^2b^2x^2y^2+(a^4e^2+b^2x^2-a^2e^2x^2)^2=a^4b^2y^2$.

4. (9), (11), (13) meet in $x=\frac{a^2b^2e^2-a^2e^2v^2-a^2v^2}{b^2u}$, $y=v$.

$\therefore u=\frac{a^2b^2e^2-a^2e^2y^2-a^2y^2}{b^2x}$, $v=y$.

The locus of this point is $a^2(b^2e^2-e^2y^2-y^2)^2+b^2x^2y^2=b^4x^2$.

5. (2), (8), (14) meet in $x=\frac{e^2u^3}{a^2}$, $y=\frac{a^2b^2+a^2e^2u^2-e^2u^4}{a^2v}$.

$\therefore u=\sqrt[3]{\frac{a^2x}{e^2}}$, $v=\left(b^2+ae\sqrt[3]{\frac{ax^2}{e}}-x\sqrt[3]{\frac{a^2x}{e^2}}\right)/y$.

Let $x=z^3$, $a/e=c^3$, then the locus of this point is $b^2c^4y^2z^2+a^2(b^2+acez^2-c^2z^4)^2=a^2b^2y^2$.

6. (4), (6), (14) meet in $x=\dfrac{a^2b^4-a^2b^2e^2v^2+a^2e^2v^4}{b^4u}$, $y=-\dfrac{a^2e^2v^3}{b^4}$.

$\therefore\ u=\left(a^2-ae\sqrt[3]{\dfrac{y^2}{ae}}+y\sqrt[3]{\dfrac{y}{a^2e^2}}\right)/x,\ v=-b\sqrt[3]{\dfrac{y}{a^2e^2}}$.

Let $y=z^3$, $1/ae=c^3$. $\therefore\ u=(a^2-aecz^2+c^2z^4)/x$, $v=-bc^2z$.
The locus of this point is $(a^2-acez^2+c^2z^4)^2+a^2c^4x^2z^2=a^2x^2$.
The loci of 5 and 6 can be found in terms of x and y by expansion.

7. (7), (12), (16) meet in $x=u$, $y=\dfrac{a^2e^2(a^2-u^2)(e^2u^2-a^2)-b^4u^2}{a^4v}$.

$\therefore\ u=x$, $v=(a^4e^4x^2-a^2e^4x^4-a^6e^2+a^4e^2x^2-b^4x^2)/a^4y$.
The locus of this point is $a^6b^2x^2y^2+(a^4e^4x^2-a^2e^4x^4-a^6e^2+a^4e^2x^2-b^4x^2)^2=a^8b^2y^2$.

8. (7), (13), (15) meet in $x=\dfrac{a^2(a^2b^2e^4v^2-a^2e^4v^4-a^2b^2v^2+b^6e^2-b^4e^2v^2)}{b^6u}$

$y=v$. $\therefore\ u=\dfrac{a^2(a^2b^2e^4y^2-a^2e^4y^4-a^2b^2y^2+b^6e^2-b^4e^2y^2)}{b^6x}$, $v=y$.

The locus of this point is $a^2(a^2b^2e^4y^2-a^2e^2y^4-a^2b^2y^2+b^6e^2-b^4e^2y^2)^2+b^{10}x^2y^2=b^{12}x^2$.

9. (3), (6), (7) meet in $x=\dfrac{b^2e^2u^4+a^4v^2}{a^2b^2u}$, $y=-\dfrac{b^4v+a^2e^2v^3}{b^4}$.

Also, $x=\dfrac{a^2b^4e^2-2a^2b^2e^2v^2+a^2e^2v^4+a^2b^2v^2}{b^4u}$.

10. (5), (7), (8) meet in $x=-\dfrac{a^2u-e^2u^3}{a^2}$, $y=\dfrac{b^6u^2-a^4e^2v^4}{a^2b^4v}$.

Also, $y=\dfrac{b^2u^2-a^4e^2+2a^2e^2u^2-e^2u^4}{a^2v}$.

11. (17), (18), (19) meet in $x=\dfrac{b^4u^3}{a^4v^2-b^4u^2}$, $y=-\dfrac{a^4v^3}{a^4v^2-b^4u^2}$.

Also, $x=\dfrac{b^2u^3}{a^4-(a^2+b^2)u^2}$, $y=\dfrac{a^2v^3}{b^4-(a^2+b^2)v^2}$.

The loci of 9, 10, 11 can be found by solving cubic equations or by means of Calculus. In case 11, it may also be solved as follows:

$$x/y=-\frac{b^4u^3}{a^4v^3} \text{ or } u/v=-\frac{a}{b}\sqrt[3]{\frac{ax}{by}}. \quad \therefore u=-\frac{av}{b}\sqrt[3]{(ax/by)}, \quad v=-\frac{bu}{a}\sqrt[3]{(by/ax)},$$

from values of x, y we get $u+x=\frac{a^2x}{b^2}\sqrt[3]{\frac{b^2y^2}{a^2x^2}}$, $v+y=\frac{b^2y}{a^2}\sqrt[3]{\frac{a^2x^2}{b^2y^2}}$.

These relations also apply to the hyperbola.

ON CERTAIN PROOFS OF THE FUNDAMENTAL THEOREM OF ALGEBRA.

By DR. ROBERT E. MORITZ, Assistant Professor in the University of Nebraska.

I.

Until an American text-book on Higher Algebra shall appear, the great majority of American students will probably continue to approach the Fundamental Theorem of Algebra through the well known English texts of Chrystal, Burnside and Panton, or Todhunter. It is therefore to be regretted that these texts, though they aim at considerable rigor in demonstration, fail when it comes to this most important theorem. Yet it is this very theorem where rigor means all, for the mere fact which the theorem embodies, is well known to every student long before he reaches the demonstration in question. It is my purpose to point out as briefly as possible certain of these tacit assumptions employed by the several authors, in the hope that if it is not desirable to entirely avoid them, as has been done by Weber in his classic text on algebra, they may at least be explicitly stated as such, in future editions of these and other texts.

II.

Chrystal's* proof is in outline as follows: To prove that one value of z, in general a complex number, can always be found which causes the rational integral function, $f(z)$, to vanish, "we have to show that a value of z can always be found which shall render mod $f(z)$ smaller than any assignable quantity. This will be established if we can show that however small mod $f(z)$ be, provided it be not zero, we can always, by properly altering z, make mod $f(z)$ smaller still." The proof now consists in showing that so long as $f(z)\neq 0$, an increment h of z may be so determined that

$$\text{mod } f(z+h) < \text{mod } f(z);$$

*Chrystal, *Algebra*, I, Chapter XII, §22.

that is, so long as mod$f(z)$ is not zero, it may be diminished, and hence it is concluded that one value of z can be found such that mod$f(z)=0$, that is, such that $f(z)=0$.

The assumption here is, that if for every value of $f(z)$ an h can be found such that mod $f(z+h)<$mod $f(z)$, $f(z)$ must necessarily vanish for some value of z. This is a non sequitur. The inference is not warranted that a function which permits of diminution for every value of the argument possesses necessarily a zero value. If, for example,

$$f(x)=\left(\frac{1+x^2}{x^2}\right)^{x^2+1},$$

$f(x)$ has no zero value, yet for every value of x an h may be found such that

$$f(x+h)<f(x).$$

III.

Burnside and Panton* retain through successive editions of their work the following proof: Suppose $z=x+iy$ is represented by the point (x, y) in the z-plane and its image $w=f(x+iy)=u+iv$ by the point (u, v) in the w-plane. If a point $z_1=z+h$, sufficiently near z, is made to describe a small closed curve about z, its image $w_1=w+k$ will describe a corresponding small closed curve about w. Suppose, now, that there is no value of z which makes $f(z)=0$, then there must be some point z, whose image w is nearer the origin than that of any other point z. But by properly selecting the path of z_1, the path of its image w_1 can be made to pass between w and the origin. This is contrary to the supposition that w is the nearest possible position with reference to the origin, consequently no value different from zero can be the least possible value of mod $f(z)$.

The unwarranted assumption in this proof is, that there must be some z whose image w is necessarily nearer the origin than that of any other z. Nor is the objection removed by letting w be the image nearer the origin (or at least as near as) the image of any other point z, as has been done by Fine† in his otherwise admirable proof of this theorem. The real objection is that w is assumed to possess a nearest position to 0 at all, that is, that the function mod $f(z)$ is assumed to possess a minimum, a supposition the contrary of which is certainly conceivable and may be legitimately held so long as nothing to the contrary has been established. In fact, it is possible to construct an algebra in which the assumption does not hold. We need only to confine ourselves to the domain of real numbers which are greater than unity. Let x be any number of this domain, then

$$f(x)=\frac{x^3+2x}{1+2x^2}$$

*Burnside and Panton, *Theory of Equations*, Fourth Edition, Vol. I, §123.
†Fine, *The Number-system of Algebra*, §54.

has no minimum value in this domain. For so long as x is real and less than unity, $f(x)$ also is real and less than unity, and

$$f(x)<x,$$

that is, for every conceivable value of $f(x)$ of our domain we can construct another which is less.

IV.

Todhunter,* in spite of his extreme caution in dealing with demonstrations of a critical nature, also assumes the existence of a minimum without recognizing it. He himself outlines his argument in the words, "Since U^2+V^2 (that is, mod $f(z)$) is always a real positive quantity, if it can not be zero there must be some value which can not be diminished; but we shall now prove that if U^2+V^2 have any value different from zero we can diminish that value by a suitable change in the expression which is substituted for x, so that it follows that U^2+V^2 must be capable of the value zero, that is, U and V must vanish simultaneously." The argument as here stated really involves two unwarranted assumptions:

1. That the function U^2+V^2 necessarily possesses a minimum,
2. That a function which permits of being indefinitely diminished must necessarily approach the limit zero.

The second of these assumptions is removed in a postscript article, but the first is ignored.

It may be mentioned in conclusion that the assumption of the existence of a minimum occurs in the common source of the various proofs which I have cited, the so-called first proof of Cauchy,† or going back still further in Legendre‡ who first developed the essential principles of Cauchy's proof.

THE UNIVERSITY OF NEBRASKA, *May, 1903.*

A GENERAL NOTATION FOR VECTOR ANALYSIS.

By JOSEPH V. COLLINS, Stevens Point, Wis.

Vector analysis is now a little over half a century old. As compared with most other branches of mathematical thought its cultivation is very recent. As we look back over the history of the development of the mathematical notations we see that at first there was great diversity in the signs used, but that this diversity in time gave place to uniformity.

*Todhunter, *Theory of Equations*, Chapter II.
†A. Cauchy, *Course d'Analyse*, Chapter XII, §1, Theorem I.
‡Legendre, *Theorie des Numbers*, 1re Part, §XIV; In the German translation by Maser, §119.

As matters now stand the several branches of vector analysis have each a separate notation. Thus there is the S and V notation of quaternions, the hooked bracket and sign of the complement of Grassmann's system, and Professor Gibbs's dot and cross, all denoting, if not always the same concepts, at least the same numerical quantities. Professor Gibbs's notation is distinctly an innovation and marks a tendency away from uniformity.

It is easy to infer why Professor Gibbs desired to throw overboard the quaternion notation. In the first place he desired juxtaposition to denote his so-called indeterminate products and then he wanted the places before and after his dyads unoccupied so that any new sign of operation desired could be inserted. This the quaternion S and V notation did not admit of. Again Professor Gibbs regards S and V as product rather than function symbols so that writing Sab and Vab is to him like writing $\times$ab for a$\times$b. It is evident from this that the quaternion notation lacks symmetry. But there is still another objection to S and V as operation symbols. They are large and for that reason do not look well, nor can they be made quickly. Presumably it was for these reasons that Professor Gibbs gave up the quaternion notation entirely and introduced the dot and cross, the first to take the place of S and the latter of V.

Professor Gibbs has used a marked improvement in the general notation for vector analysis along another line in employing Clarendon type to denote vectors and the same letters in italics to denote their tensors. He thus avoids the use of Hamilton's cumbrous T and U. His system however is open to serious criticism. Besides being out of harmony with the quaternion symbols, this notation contains an important defect in that it uses up all of the signs available to signify multiplication, viz., juxtaposition, the dot, and the cross. Now there are other forms of product formation besides the dyadic and scalar and vector products, as for instance, the quaternion itself, or a product in which enters a trigonometrical function other than the sine and cosine. To have a system of notation to denote other forms of product, one would be under the necessity of inventing for them new arbitrary marks, which course would be open to serious objection. Fortunately all interests can be harmonized and ends met by the simple expedient of using small *letters* for the operation symbols and writing them *between* their factors.

The complete notation proposed may be described very briefly as the use of capitals to denote unit quantities, Clarendon type for vector quantities, the corresponding letters in italics for their tensors, and small letters, preferably initials, written a little above the line of writing, as symbols of operation, with the single exception of the indeterminate product, where no sign is used. Of course between two scalars or between a scalar and a vector no sign means multiplication. Thus ab=dyad or indeterminate product of vectors a and b.

$a^{s}b$=Sab=scalar product of vectors a and b.

$a^{V}b$=Vab=vector product of vectors a and b.

$a^{t}b$=scalar product of tensors of a and b times tangent of included angle.

aq b=quaternion product of vectors a and b.

aq b, or [ab]=Grassmann outer product of vectors a and b.

The writer has found this notation as convenient as the dot and cross. It is commended to the student of vector analysis as practical and as retaining the original Hamilton symbols. Evidently little effort is needed to pass from this to the old S, V, T, U notation. Finally, it is adapted to the needs of the general science of vector analysis and admits freely of the introduction of new products.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

167. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A traveler notices that $m=2\frac{1}{2}$ times the number of spaces between the telegraph poles that he passes in a minute is the rate of the train in miles per hour. How far apart are the poles?

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let n=number of spaces passed per minute, d=distance between poles.

Then mn miles per hour $=\frac{mn\times 5280}{60}=88mn$ feet per minute.

$\therefore dn=88mn$, or $d=88m$ feet. When $m=2\frac{1}{2}$, $d=220$ feet.

168. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If I buy for $m=10\%$ and $n=5\%$ *less* I shall gain $p=15\%$ and $q=5\%$ *more*. What is my rate of gain?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This problem is not clear. I interpret it to mean: If I buy for $100\times 90\times 95=85.5\%$ instead of for 100%, and sell for the same price as I sold when I paid 100%, I would gain $100\times 115\times 105=120\frac{3}{4}\%$, $-100\%=20\frac{3}{4}\%$ more than I did gain. $85\frac{1}{2}\%=\frac{171}{200}$, $20\frac{3}{4}\%\cdot\frac{83}{400}$, $\frac{83}{400}+1=\frac{483}{400}$. $(\frac{483}{400}\times 171-200)\div(200-171=(\frac{82593}{400}-200)\div(200-171)=6\frac{193}{400}\div 29=\frac{2593}{11600}$, $\frac{2593}{11600}$ of $100\%=22\frac{41}{116}\%$.

ALGEBRA.

175. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the conditions that $\frac{x}{m+3}+\frac{y}{m+1}+\frac{z}{m-z}=1$, where m may be a, b or c.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x=(m+3)u$, $y=(m-1)v$, $z=(m-z)w$.

$\therefore u+v+w=1$. $\therefore u$, v, w are the areal coördinates of a point. Let d, e, f be the sides of the triangle of reference; then

$$\frac{u}{d\alpha}=\frac{v}{e\beta}=\frac{w}{f\gamma}=\frac{1}{2\triangle},$$

where $\triangle$ = area of triangle of reference, and $d\alpha+e\beta+f\gamma=2\triangle$.

$$\therefore u=d\alpha/2\triangle,\ \therefore x=\frac{(m+3)d\alpha}{2\triangle};$$

$$v=e\beta/2\triangle,\ \therefore y=\frac{(m-1)e\beta}{2\triangle};$$

$$w=f\gamma/2\triangle,\ \therefore z=\frac{(m-z)f\gamma}{2\triangle},\ \text{or}\ z=\frac{mf\gamma}{2\triangle+f\gamma}.$$

$$\therefore \frac{x}{m+3}+\frac{y}{m-1}+\frac{z}{m-z}=\frac{d\alpha+e\beta+f\gamma}{2\triangle}=\frac{2\triangle}{2\triangle}=1,\ \text{whatever the value of } m.$$

176. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve $x^2+y^2+z=a$.....(1), $x+y^2+z^2=b$.....(2), $x^2+y+z^2=c$.....(3).

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2+y^2+z^2=s$.

$\therefore z^2-z=s-a$, or $z=\frac{1}{2}\pm\sqrt{(s-a+\frac{1}{4})}=\frac{1}{2}\pm\sqrt{(m-a)}$.

$x^2-x=s-b$, or $x=\frac{1}{2}\pm\sqrt{(s-b+\frac{1}{4})}=\frac{1}{2}\pm\sqrt{(m-b)}$.

$y^2-y=s-c$, or $y=\frac{1}{2}\pm\sqrt{(s-c+\frac{1}{4})}=\frac{1}{2}\pm\sqrt{(m-c)}$.

$\therefore x^2+y^2+z^2=s=\frac{3}{4}\pm[\sqrt{(m-a)}+\sqrt{(m-b)}+\sqrt{(m-c)}]+3m-(a+b+c)$.

$\therefore 2m+1-(a+b+c)=\mp[\sqrt{(m-a)}+\sqrt{(m-b)}+\sqrt{(m-c)}]$.

$\therefore 2m+D\pm\sqrt{(m-a)}=\mp[\sqrt{(m-b)}+\sqrt{(m-c)}]$.....(1). Squaring (1),

$$4m^2+(4D-1)m+D^2+b+c-a=2\{\sqrt{[(m-b)(m-c)]}\mp(2m+D)\sqrt{(m-a)}\}.....(2).$$

Squaring (2) and remembering that $a+b+c=1-D$,

$$16m^4+8(4D-3)m^3+(24D^2-32D+5)m^2+(8D^3-14D^2+6D+2)m$$
$$+2D^2-2D^3-2(ab+ac+bc)=\mp 8(2m+8)\sqrt{[(m-a)(m-b)(m-c)]},$$

or $16m^4+Am^3+Bm^2+Cm+E=\mp 8(2m+8)\sqrt{[(m-a)(m-b)(m-c)]}$.....(3).

Squaring (3), $256m^8+32Am^7+(A^2+32B)m^6+(32C+2AB-256)m^5$
$+[B^2+32C+2AC-2048+256(a+b+c)]m^4+[2AC+2BC$
$-4096+2048(a+b+c)-256(ab+ac+bc)]m^3+[C^2+2BE$
$+4096(a+b+c)-2048(ab+bc+ac)+256abc]m^2+[2CE$
$-4096(ab+ac+bc)+2048abc]m+E^2+4096abc=0.$

This equation gives m and hence s, which finally gives x, y, z.

Solved in a similar manner by the *PROPOSER*.

177. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve $m^{2x}(m^2+1)=(m^{3x}+m^x)m$.

Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.; CHARLES E. BASSETT, Central University, Danville, Ky.; and E. L. SHERWOOD, Shady Side Academy, Pittsburg, Pa.

The equation may be written $\dfrac{m^{3x}+m^x}{m^{2x}}=\dfrac{m^2+1}{m}$, or $m^x+\dfrac{1}{m^x}=m+\dfrac{1}{m}$,

whence $m^{2x}-\left(m+\dfrac{1}{m}\right)m^x=-1$, from which $m^x=m$ and $1/m$.

Therefore $m^{x\pm 1}=1$ and $x\pm 1=0$; $x=+1$ and -1.

Also solved by *G. W. GREENWOOD*, and *G. B. M. ZERR*. Professor Zerr finds by performing the indicated operations and factoring, in addition to the roots given above, the root $-\infty$.

GEOMETRY.

197. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Two points P_1, Q_1 are on a generator of a hyperboloid, and P_2, Q_2 the corresponding points on a confocal hyperboloid. Prove $P_1Q_1=P_2Q_2$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2/a^2-y^2/b^2-z^2/c^2=x^2/a^2-y^2/\beta^2-z^2/\gamma^2=1$ be the hyperboloid and its confocal;

P_1, $Q_1=(d, e, f)$, (h, k, l);

P_2, $Q_2=(m, n, p)$, (r, s, t). We are to prove,

$$(d-h)^2+(e-k)^2+(f-l)^2=(m-r)^2+(n-s)^2+(p-t)^2.$$

For corresponding points,

$$d/a=m/\alpha,\ e/b=n/\beta,\ f/c=p/\gamma,\ h/a=r/\alpha,\ k/b=s/\beta,\ l/c=t/\gamma.$$

$$\therefore\ [(m\alpha/\alpha)-h]^2+[(nb/\beta)-k]^2+[(pc/\gamma)-l]^2$$
$$=[m-(h\alpha/\alpha)]^2+[n-(k\beta/b)]^2+[p-(l\gamma/c)]^2,$$

or $(m/\alpha-h/a)^2(a^2-\alpha^2)+(n/b-k/b)^2(b^2-\beta^2)+(p/\gamma-l/c)^2(c^2-\beta^2)=0.$

This is clearly the case since

$$h^2/a^2-k^2/b^2-l^2/c^2=m^2/\alpha^2-n^2/\beta^2-p^2/\gamma^2=1,$$
$$\text{or } m^2/\alpha^2-h^2/a^2-(n^2/\beta^2-k^2/b^2)-(p^2/\gamma^2-l^2/c^2)=0.$$

$$\therefore\ (m/\alpha-h/a)(m/\alpha+h/a)-(n/\beta-k/b)(n/\beta+k/b)-(p/\gamma-l/c)(p/\gamma+l/c)=0.$$

This is the case when $m/\alpha=h/a,\ n/\beta=k/b,\ p/\gamma=l/c.$

198. Proposed by JOHN J. QUINN, Professor of Mathematics, Warren High School, Warren, Pa.

Trisect an angle (1) by means of the cissoid; (2) by means of the paraboloid.

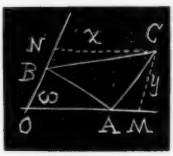
No solution of this problem has been received.

199. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Two vertices of a given triangle move along fixed right lines; find the locus of the third vertex. [From Salmon's Conics, Sixth Edition, page 208, Ex. 10.]

Solution by M. W. HASKELL, Professor of Mathematics, The University of California, Berkeley, Cal.

Let us take the fixed lines as the coördinate axes, and denote the angle between them by ω; the interior angles of the triangle by A, B, C, and the lengths of the opposite sides by a, b, c, respectively; and let φ denote the variable angle OAB. From the triangles AMC, BNC, we have immediately from the theorem of sines



$$y/b=\frac{\sin(\pi-A-\varphi)}{\sin(\pi-\omega)}=\frac{\sin(A+\varphi)}{\sin\omega};\quad x/a=\frac{\sin(\omega-B+\varphi)}{\sin\omega}$$

which may immediately be rewritten in the form

$$\sin A\cos\varphi+\cos A\sin\varphi=(y/b)\sin\omega,$$
$$\sin(\omega-B)\cos\varphi+\cos(\omega-B)\sin\varphi=(x/a)\sin\omega.$$

Solving these equations for $\cos\varphi$ and $\sin\varphi$, and observing that $\sin(A+B-\omega)=\sin(C+\omega)$, we have

$$\cos\varphi\sin(C+\omega)=\sin\omega[-(x/a)\cos A+(y/b)\cos(\omega-B)]$$
$$\sin\varphi\sin(C+\omega)=\sin\omega[(x/a)\sin A-(y/b)\sin(\omega-B)].$$

Squaring and adding, and transposing the members of the equation, we obtain the required equation of the locus

$$\frac{x^2}{a^2}+\frac{2xy}{ab}\cos(C+\omega)+\frac{y^2}{b^2}=\frac{\sin^2(C+\omega)}{\sin^2\omega}.$$

The locus is therefore an ellipse with center at the intersection of the two lines.

Solved similarly by *G. W. GREENWOOD*, and *G. B. M. ZERR*.

CALCULUS.

163. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Can there be a plane curve the length of which varies *directly as the abscissa* and *inversely as the ordinate* of any point on the curve?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$S=mx/y,\ dS=\frac{mydx-mxdy}{y^2}=\sqrt{(dx^2+dy^2)}.$$

$\therefore \sqrt{[1+(dx/dy)^2]}=[my(dx/dy)-mx]/y^2.$
Let $x=vy$, $dx/dy=v+y(dv/dy)=v+yp$, suppose.
$\therefore \sqrt{[1+(v+yp)^2]}=mp$, or $v+yp=\sqrt{(m^2p^2-1)}$.....(1).

From (1), $p=\frac{vy\pm\sqrt{(v^2m^2+m^2-y^2)}}{m^2-y^2}=\frac{xy\pm\sqrt{[m^2(x^2+y^2)-y^4]}}{y(m^2-y^2)}$.

Differentiating (1) with respect to y,

$$dv/dy=p=p--y(dp/dy)+\frac{m^2p(dp/dy)}{\sqrt{(m^2p^2-1)}}.$$

$$\therefore 2p=\left(\frac{m^2p}{\sqrt{(m^2p^2-1)}}-y\right)\frac{dp}{dy}.$$

$$\therefore \frac{dy}{dp}=\frac{m^2}{2\sqrt{(m^2p^2-1)}}-\frac{y}{2p} \text{ or } \frac{dy}{dp}+\frac{y}{2p}=\frac{m^2}{2\sqrt{(m^2p^2-1)}}.$$

$$\therefore y\sqrt{p}=C+\frac{m^2}{2}\int\frac{\sqrt{(p)}dp}{\sqrt{(m^2p^2-1)}}=C+\frac{m^2}{2}f(p).....(2).$$

The value of p in (2) from (1) gives the primitive, which is the curve desired. By series we get

$$\frac{m^2}{2}\int\frac{\sqrt{(p)}dp}{\sqrt{(m^2p^2-1)}}=\frac{1}{2\sqrt{p}}\sqrt{(m^2p^2-1)}+\tfrac{1}{2}m\sqrt{(p)}\left(1+\frac{1}{2.3m^2p^2}+\frac{1}{2.4.7}\frac{1}{m^4p^4}\right.$$

$$\left.+\frac{3}{2.4.6.11}\frac{}{m^6p^6}+\frac{3.5}{2.4.6.8.15m^8p^8}+\ldots\ldots\right).$$

164. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $m^2+n^2=1$, $m^2\cos^2\theta+n^2\cos^2\varphi=A$, $a^2b^2\sin^2\theta(m^2+n^2\cos^2\varphi)+a^2c^2\cos^2\theta\cos^2\varphi+b^2c^2\sin^2\varphi(n^2+m^2\cos^2\theta)=B$, $\sqrt{(1-m^2\sin^2\theta)}=\triangle(\theta)$, $\sqrt{(1-n^2\sin^2\varphi)}=\triangle(\varphi)$ prove that $\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{\triangle(\theta)\triangle(\varphi)}=(\tfrac{1}{8}\pi)(a^2b^2+a^2c^2+b^2c^2)$.

Solution by the PROPOSER.

$x^2/a^2+y^2/b^2+z^2/c^2=1$.

$$\int ds=\int\int\sqrt{[1+(dz/dx)^2+(dz/dy)^2]}dxdy=c^2\int\int\frac{\sqrt{[z^2/c^4+x^2/a^4+y^2/b^4]}dxdy}{z}$$

Let p=perpendicular from center on tangent plane.

$\therefore\ 1/p=\sqrt{[z^2/c^4+x^2/a^4+y^2/b^4]}$.

Let $(b/a)\sqrt{[a^2-x^2]}=y'$, $(a/b)\sqrt{[b^2-y^2]}=x'$.

$$D=\int\frac{ds}{p}=c^2\int\int\frac{[z^2/c^4+x^2/a^4+y^2/b^4]dxdy}{z}=\frac{1}{c^2}\int_0^a\int_0^{y'}zdxdy+\frac{c^2}{a^4}\int_0^a\int_0^{y'}\frac{x^2}{z}dxdy$$

$$+\frac{c^2}{b^4}\int_0^b\int_0^{x'}(y^2/z)dydx=\frac{1}{abc}\int_0^a\int_0^{y'}\sqrt{[a^2b^2-b^2x^2-a^2y^2]}dxdy$$

$$+\frac{abc}{a^4}\int_0^a\int_0^{y'}\frac{x^2dxdy}{\sqrt{[a^2b^2-b^2x^2-a^2y^2]}}+\frac{abc}{b^4}\int_0^b\int_0^{x'}\frac{y^2dydx}{\sqrt{[a^2b^2-b^2x^2-a^2y^2]}}$$

$$=\tfrac{1}{6}\pi(ab/c+bc/a+ac/b)=\frac{\pi}{6abc}(a^2b^2+b^2c^2+a^2c^2).$$

Let $x=a\sin\varphi\sqrt{(1-m^2\sin^2\theta)}$, $y=b\cos\theta\cos\varphi$, $z=c\sin\theta\sqrt{(1-n^2\sin^2\varphi)}$, $m^2+n^2=1$; $dx/d\varphi=a\cos\varphi\sqrt{(1-m^2\sin^2\theta)}$, $dx/d\theta=-\frac{am^2\sin\varphi\sin\theta\cos\theta}{\sqrt{(1-m^2\sin^2\theta)}}$, $dy/d\varphi=-b\cos\theta\sin\varphi$, $dy/d\theta=-b\sin\theta\cos\varphi$; $dxdy=(dx/d\theta)(dy/d\varphi)-(dx/d\varphi)(dy/d\theta)$

$$=\frac{ab}{\sqrt{(1-m^2\sin^2\theta)}}(m^2\sin^2\varphi\sin\theta\cos^2\theta+\sin\theta\cos^2\varphi-m^2\sin^3\theta\cos^2\varphi)=\frac{Aab\sin\theta}{\triangle(\theta)}.$$

$$\frac{c^2}{z}(x^2/a^4+y^2/b^4+z^2/c^4)=\frac{c}{\sin\theta\,\triangle(\varphi)}\left(\frac{\sin^2\varphi(1-m^2\sin^2\theta)}{a^2}+\frac{\cos^2\theta\cos^2\varphi}{b^2}\right.$$

$$\left.+\frac{\sin^2\theta(1-n^2\sin^2\varphi)}{c^2}\right)=\frac{1}{a^2b^2c\sin\theta\,\triangle(\varphi)}[b^2c^2\sin^2\varphi(n^2+m^2\cos^2\theta)$$

$$+a^2c^2\cos^2\theta\cos^2\varphi+a^2b^2\sin^2\theta(m^2+n^2\cos^2\varphi)]=\frac{B}{a^2b^2c\sin\theta\,\triangle(\varphi)}.$$

$$\therefore\int\frac{ds}{p}=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{abc\,\triangle(\theta)\,\triangle(\varphi)}=\frac{\pi}{6abc}(a^2b^2+a^2c^2+b^2c^2).$$

$$\therefore\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{\triangle(\theta)\,\triangle(\varphi)}=\tfrac{1}{6}\pi(a^2b^2+a^2c^2+b^2c^2).$$

$$\text{Also }\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{ABd\theta d\varphi}{\triangle(\theta)\,\triangle(\varphi)}=\tfrac{1}{3}(a^2b^2+a^2c^2+b^2c^2)[E(m)F(n)$$

$$+F(m)E(n)-F(m)F(n)].$$

$$\therefore E(m)F(n)+F(m)E(n)-F(m)F(n)=\tfrac{1}{2}\pi,\text{ Legendre's Theorem.}$$

MECHANICS.

155. Proposed by M. E. GRABER, Graduate Student, Heidelberg University, Tiffin, Ohio.

A parabolic curve is placed in a vertical plane with its axis vertical and vertex downwards, and inside of it, and against a peg in the focus, and against the concave arc, a smooth uniform and heavy beam rests. Find the position of equilibrium.

Solution by G. W. GREENWOOD, A. B., Professor of Mathematics, McKendree College, Lebanon, Ill., and E. L. SHERWOOD, Shady Side Academy, Pittsburg, Pa.

Let the equation to the parabola be $2a/r=1+\cos\theta$, and let the length of the rod be $2l$. We must have $2l>a$. When the rod makes an angle θ with the axis, the depth of its center of gravity below the focus is

$$n=(r-l)\cos\theta=\left(\frac{2a}{1+\cos\theta}-l\right)\cos\theta.$$

In a position of equilibrium, n is a maximum or a minimum.

$$\therefore\sin\theta\left(\frac{2a}{(1+\cos\theta)^2}-l\right)=0.\quad\therefore\theta=0,\text{ or }\cos\theta=\sqrt{(2a/l)}-1.$$

We must take the positive sign with the radical.

Since $2l>a$, $\sqrt{(2a/l)}<2$, and the values of θ obtained from the last equation are real, but in order that θ be not greater than 90°, an obvious requirement in this problem, we must have $l<2a$. In any case, the position given by $\theta=0$ is unstable. The two values of θ given by the second equation give the positions of stable equilibrium when $2a>l>\frac{1}{2}a$.

Also solved by *G. B. M. ZERR.*

156. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Eng.

Three perfectly elastic particles start from the cusp of a smooth cycloid (axis vertical, vertex down) at intervals of t seconds. How long will it be to the nth collision?

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Since the curve is a smooth cycloid and the particles perfectly elastic and of equal mass, whenever collision takes place the two particles that collide have the same velocity. The reaction of one upon the other from impact is the same as though one particle passed through the other without either being affected.

Let mt $(m>2)$ be the time it takes a particle to go from cusp to cusp, and a, the radius of the generating circle. Then $m=(2\pi/t)\sqrt{(a/g)}$. After the first particle has arrived at the opposite cusp, the second is t and the third $2t$ seconds behind it.

$\therefore$ Time to first collision is $mt+t/2=(2m+1)t/2$ seconds.
Time to second collision is $mt+t=(2m+2)t/2$ seconds.
Time to third collision is $mt+3t/2=(2m+3)t/2$ seconds.
Time to fourth collision is $mt+t+mt-t+t/2=(4m+1)t/2$ seconds.
Time of fifth collision is $(4m+2)t/2$ seconds.
Time of sixth collision is $(4m+3)t/2$ seconds; etc.

$\therefore$ The collisions take place in sets of three. The pth set is $(2pm+1)t/2$, $(2pm+2)t/2$, $(2pm+3)t/2$. If the nth collision is the 1st, 4th, 7th, ...$(3p-2)$th, then $n=3p-2$ or $p=(n+2)/3$, and it takes place in $[2(n+2)m+3]t/6$ seconds; if it is the 2nd, 5th, 8th, ...$(3p-1)$th, it takes place in $[2(n+1)m+6]t/6$ seconds; if it is the 3rd, 6th, 9th, ...$3p$th, it takes place in $(2nm+9)t/6$ seconds.

Also solved by *G. W. GREENWOOD.*

157. Proposed by T. W. WRIGHT, Professor of Mathematics, Schenectady, N. Y.

Explain why a waterfall h feet high can support a column $2h$ feet high.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let w=weight of a cubic foot of water. A column of water 1 foot square falling through h feet weighs hw pounds; to this must be added the kinetic energy of a cubic foot of water falling through h feet; this equals $wv^2/2g$, but $v^2=2gh$.

$\therefore$ kinetic energy$=wh$. $\therefore$ $wh+wh=2wh=w(2h)$.

158. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are electrical images of each other in respect to a given sphere may be constant, and that their centers of gravity shall lie on the surface of the sphere.

Remark by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This problem is the same as problem 148, a solution of which is given in No. 2, Vol. X, pages 48 and 49.

DIOPHANTINE ANALYSIS.

105. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that every factor of $a^{2m}+b^{2m}$ is of the form $1[\text{mod}.2^{m+1}]$ where a and b are prime to each other.

Note by the PROPOSER.

The enunciation of this proposition is corrected on page 207, Vol. IX, of the MONTHLY, and it may be stated in full as follows:

If a, b and m are real positive integers, a and b relatively prime, then every odd factor of $a^{2m}+b^{2m}$ has the form $1\{mod.2^{m+1}\}$.

This special arithmetical theorem was first given by Euler, and may be directly proved by the principles in the theory of power residues. It is interesting, however, to examine its connection with a more general proposition. If $\omega_1, \omega_2 \ldots\ldots \omega_{\phi(n)}$ are all the distinct primitive roots of $x^n-1=0$ and $\phi(n)$ the number of integers less than the positive integer n, and prime to it, then the rational function

$$F\{a, b, n\}=\{a-\omega_1 b\}\{a-\omega_2 b\}\ldots\ldots\{a-\omega_{\phi(n)} b\}$$

possesses the following property:

Every divisor of $F\{a, b, n\}$ which is prime to n, has the form $1\{mod\, n\}$.

A proof of this for the case $b=1$, is to be found in Kronecker's *Vorlesungen über Zahlentheorie*," Vol. I, pages 440-441, and the analysis used may be readily extended to cover the case in which b is general. For $n=2^{m+1}$ (m, a positive integer) it will be noticed that

$$F\{a, b, 2^{m+1}\}=a^{2m}+b^{2m},$$

and by the general theorem above stated, it follows that every odd divisor of this function has the form $1(\text{mod}2^{m+1})$. When $a=2$, $b=1$, we may write

$$f\{m\}=2^{2^m}+1,$$

and this form gives the remarkable numbers of Gauss which occur in the partition of the perigon.

I notice that the determination of factors of $f\{m\}$ for m=9, 11, 12, 18 and 38, has been recently effected by several English mathematicians. For information regarding these interesting results, see the *Educational Times*, June, 1903, page 270, where reference is made to the announcement by Col. J. A. Cunningham. A summary of the previous work done along this line in given in Ball's *Recreations and Problems*, third edition, pages 34-35.

112. Proposed by L. C. WALKER, A. M., Graduate Student Leland Stanford Jr. University, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5, the first triangle, find the sides of the nth triangle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and the PROPOSER.

Let $2x_n-1$, $2x_n$, and $2x_n+1$ denote the sides of the nth triangle, r_n the radius of the inscribed circle, and A the area of the triangle.

Then $A=\sqrt{\{[3x_n(x_n+1)(x_n)(x_n-1)]\}}=3x_nr_n$,

whence $[x_n-r_n\sqrt{3}][x_n+r_n\sqrt{3}]=1$.....[1].

Now for the first triangle, x=2, and r=1; $\therefore [2-\sqrt{3})^n[2+\sqrt{3}]^n=1$.....[2].

Assume $x_n-r_n\sqrt{3}=[2-\sqrt{3}]^n$; then from [1], $x_n+r_n\sqrt{3}=[2+\sqrt{3}]^n$; from which $x_n-r_n\sqrt{3}=\frac{1}{2}\{[2+\sqrt{3}]^n+[2-\sqrt{3}]^n\}$. Hence,

$$2x_n-1=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n-1,$$

$$2x_n=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n,$$

$$2x_n+1=[2+\sqrt{3}]^n+[2-\sqrt{3}]^n+1,$$

$$A=[\tfrac{1}{4}\sqrt{3}]\{[2+\sqrt{3}]^{2n}-[2-\sqrt{3}]^{2n}\}.$$

II. Solution bp A. H. BELL, Litchfield, Ill.

Take the sides of the rational triangle $x+d$, x, $x-d$; the area of this triangle is $\frac{1}{4}x[3(x^2-4d^2)]^{\frac{1}{2}}$. To have it rational we must have $3[x^2-4d^2]=\square=y^2$, say, in which the area is $\frac{1}{4}xy$. If we make $x=2x'$, and $y=6y'$, the equation becomes, when d=1, $x'^2-3y'^2=1$.....(2). The values of x, by Lagrange, are the numerators of the odd convergents from the $\sqrt{3}$; the first two odd fractions are $\frac{1}{0}, \frac{2}{1}$, etc., in which $2x'=x=4$, giving the initial rational triangle with sides 3, 4, 5.

The extension of the values of x', or x, by the algebraic solution of this equation in general, is known to be $x_{n+1}=2x.x_n-x_{n-1}$.....(4).

This constant factor $2x$, is called by Roberts, and Robins, the magic M, or Key. In this case M=4, and the middle side of the next rational triangle is by (4), $x_2=4.4-2=14$, and the triangle 13, 14, 15; by extending the middle side in the same way we get,

No. of triangle:	1	2	3	4	5	6	
Sides:	3	13	51	193	723	2701	etc.....(5).
	4	14	52	194	724	2702	
	5	15	53	195	725	2703	

The nth rational triangle can be found by entending formula (4) in terms of M, and x', as follows:

No.	0	1	2	3	4
Series	$x'=1$;	x';	$Mx'-1$;	$[M^2-1]x'-M$;	$[M^3-2m]x'-M^2+1$;

5		
$[M^4-3M^2+1]x'-M^3+2M$;	etc.	etc.....(6).

By restoring the value of x', above=2, we have the series of values of the nth, middle side of the nth rational triangle in terms of the powers of magic M. The quantity in the powers of M multiplied by the constant factor x', I call the "Continued Fraction Theorem" the laws of which are easily determined. These quantities are shown above in brackets in (6), and abbreviated to CFT.

Hence the middle side of the nth triangle $=\{[n\text{th.CFT}]2-(n-1]\text{th.CFT}\}2$.....(7).

The laws of this theorem are: The 1st coefficient of M=1.

The 2nd coefficient of M=no. cycle -2.

The pth coefficient of M=(where $n-p=m$)

$$p\text{th coefficient}=\frac{m[m-1][m-2].....[m-p+2]}{1.2.3.....[p-1]}..... (8).$$

If $n=10$, the 10th.CFT$=[M^9-8M^7+21M^5-20M^3+5M]$, and multiplied by $\frac{1}{2}M=x'=2$, from which take 9th.CFT$=M^8-7M^6+15M^4-10M^2+1$, the result multiplied by 2 will give the middle side, x, of the 10th rational triangle=524174.

Then the 10th triangle has for sides 524173, 524174, 524175.

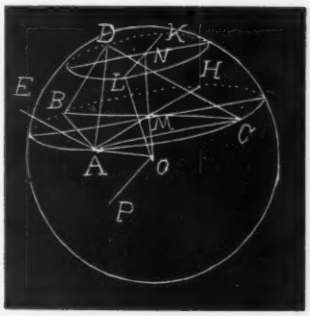
AVERAGE AND PROBABILITY.

130. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of a tetrahedron formed by the planes passing through the points taken three by three, is 1-35 of the voume of the given sphere.

II. Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B, C, D be the four random points; AH the diameter of the section through A, B, C; LK the diameter of a section through D parallel to the section through A, B, C; O the center of the sphere, M the center of AH, N the center of LK. Draw AE perpendicular to AH, and OP a line such that AB is parallel to the plane MOP. Let $AO=r$, $\angle AOM=\theta$, $\angle EAC=\varphi$, $\angle EAB=\psi$, $\angle LON=\beta$, $\angle MOP=\lambda$, the angle POM makes with some fixed plane$=\rho$.



Then $AC=2r\sin\theta\sin\varphi$, $AB=2r\sin\theta\sin\psi$; the volume, V, of the tetrahedron $DABC=\frac{2}{3}r^3(\cos\beta-\cos\theta)\sin^2\theta\sin\varphi\sin\psi\sin(\varphi-\psi)$. An element of surface at A is $2\pi r^2\sin\theta d\theta$; at D, $2\pi r^2\sin\beta d\beta$; at C, $4r^2\sin\theta\sin\varphi d\varphi d\lambda$; at B, $4r^2\sin\theta\sin(\varphi-\psi)\sin\lambda\sin\psi d\psi d\rho$. The limits of θ are 0 and π; of β, 0 and θ; of φ, 0 and π; of ψ, 0 and φ and doubled; of λ, 0 and π; of ρ, 0 and 2π. Since the whole number of ways the four points can be taken is $256\pi^4r^8$, we have

$$\triangle = \frac{2}{256\pi^4 r^8}\int_0^{\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{\phi}\int_0^{\pi}\int_0^{2\pi} V.2\pi r^2\sin\theta d\theta.2\pi r^2\sin\beta d\beta.4r^2\sin\theta\sin\varphi d\varphi d\lambda$$

$$\times 4r^2\sin\theta\sin(\varphi-\psi)\sin\lambda\sin\psi d\psi d\rho$$

$$= \frac{2r^3}{3\pi}\int_0^{\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{\phi}\int_0^{\pi}\sin^5\theta(\cos\beta-\cos\theta)\sin\beta\sin^2\varphi\sin^2\psi\sin^2(\varphi-\psi)\sin\lambda d\theta d\beta$$

$$\times d\varphi d\psi d\lambda$$

$$= \frac{4r^3}{3\pi}\int_0^{\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{\phi}\sin^5\theta\sin\beta(\cos\beta-\cos\theta)\sin^2\varphi\sin^2\psi\sin^2(\varphi-\psi)d\theta d\beta d\varphi d\psi$$

$$= \frac{r^3}{6\pi}\int_0^{\pi}\int_0^{\theta}\int_0^{\pi}\sin^5\theta\sin\beta(\cos\beta-\cos\theta)(3\varphi\sin^2\varphi-2\varphi\sin^4\varphi-3\sin^5\varphi\cos\varphi)d\theta d\beta d\varphi$$

$$= \frac{r^3}{16\pi}\int_0^{\pi}\int_0^{\theta}\sin^5\theta\sin\beta(\cos\beta-\cos\theta)d\theta d\beta$$

$$= \frac{\pi r^3}{32}\int_0^{\pi}\sin^5\theta(2-2\cos\theta-\sin^2\theta)d\theta = (4\pi r^3/105) = \tfrac{1}{35} \text{ of volume of sphere.}$$

133. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

A circle of unknown radius is drawn with its center at the vertex of a given parabola, and has its greatest area when its circumference passes through the focus of the parabola. Required the average area common to the circle and parabola.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $y^2=4ax$, $x^2+y^2=r^2$ be the equation to the parabola and circle, respectively, $\triangle$ =average area, $x'=\sqrt{(r^2+4a^2)}-2a$.

$$\therefore \triangle = 2\int_0^a \left[2\int_0^{x'}\sqrt{(ax)}dx+\int_{x'}^{r}\sqrt{(r^2-x^2)}dx\right]dr/\int_0^a dr$$

$$= \int_0^a \left[2\int_0^{x'}\sqrt{(ax)}dx+\int_{x'}^{r}\sqrt{(r^2-x^2)}dx\right]dr = \frac{1}{6a}\int_0^a\Big[3\pi r^2+4\sqrt{(a)}[\sqrt{(r^2+4a^2)}$$

$$-2a]^{\frac{3}{2}}-6r^2\sin^{-1}\left(\frac{\sqrt{(r^2+4a^2)}-2a}{r}\right)\Big]dr = \tfrac{1}{6}\pi a^2-\tfrac{1}{3}a^2\sin^{-1}(\sqrt{5}-2)$$

$$+\frac{2\sqrt{a}}{3a}\int_0^a[\sqrt{(r^2+4a^2)}-2a]^{\frac{3}{2}}dr+\frac{\sqrt{a}}{3a}\int_0^a\frac{r^2[\sqrt{(r^2+4a^2)}-2a)^{\frac{1}{2}}dr}{\sqrt{(r^2+4a^2)}}.$$

Let $\sqrt{(r^2+4a^2)}=2y-r$, $\frac{a}{2}(\sqrt{5}+1)=y'$.

$$\therefore \Delta=\tfrac{1}{6}\pi a^2-\tfrac{1}{3}a^2\sin^{-1}(\sqrt{5}-2)+\frac{\sqrt{a}}{3a}\int_a^{y'}\frac{(3y^2+3a^2+2ay)(y-a)^3dy}{y^3\sqrt{y}}$$

$$=\tfrac{1}{6}\pi a^2-\tfrac{1}{3}a^2\sin^{-1}(\sqrt{5}-2)+\frac{\sqrt{a}}{3a}\int_a^{y'}(3y\sqrt{y}-7a\sqrt{y}+6a^2/\sqrt{y}-6a^3/y\sqrt{y}$$

$$+7a^4/y^2\sqrt{y}-3a^5/y^3\sqrt{y})dy=\tfrac{1}{6}\pi a^2-\tfrac{1}{3}a^2\sin^{-1}(\sqrt{5}-2)+\tfrac{2}{45}a^2[(86-13\sqrt{5})\sqrt{\frac{\sqrt{5}+1}{2}}-128+11(11-2\sqrt{5})\sqrt{\frac{\sqrt{5}-1}{2}}].$$

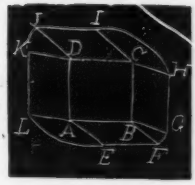
Solved with a different result by the *PROPOSER*.

134. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

An ellipse, semi-axes a, b, is placed on a square, side c. Find the chance that center of ellipse is on the square.

Solution by the PROPOSER.

Let $ABCD$ be the square, and let the ellipse move parallel to itself so that the major axis makes an angle θ with the side BC. Then the center of the ellipse will describe the area $GHIJKLEFG$. Let $(a^2-b^2)/a^2=e^2$, p=chance. The perpendicular distance from BC to $GH=a\sqrt{(1-e^2\cos^2\theta)}$, the perpendicular distance from DC to $IJ=a\sqrt{(1-e^2\sin^2\theta)}$.



Area $=\pi ab+c^2+2ac\,[\sqrt{(1-e^2\sin^2\theta)}+\sqrt{(1-e^2\cos^2\theta)}]=u$.

$$1/p=\int_0^{\frac{1}{2}\pi}ud\theta/\int_0^{2\pi}c^2d\theta=\frac{2}{\pi c^2}\int_0^{\frac{1}{2}\pi}ud\theta.\quad\therefore\ 1/p=[\pi ab+c^2+\frac{8ac}{\pi}E(e,\tfrac{1}{2}\pi)]/c^2.$$

If $a=b$, $1/p=(\pi a^2+c^2+4ac)/c^2$. If $a=b=c$, $1/p=\pi+1+4$.

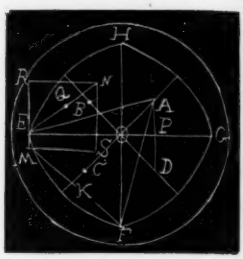
135. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

If the line joining two points taken at random in the surface of a given circle be the diagonal of a square, the chance that the square lies wholly within the circle is $2-4/\pi$.

Solution by the PROPOSER.

Let MN be the line joining the random points M, N; $MRNS$ the square; Q, its center; O, the center of the circle. Let the square move about the circle, so as to be within it, but in contact with it, the diagonal MN remaining parallel

to itself; Q will describe the figure $EFGH$, whose boundary consists of the four equal arcs, EF, FG, GH, HE, whose centers are A, B, C, D, and radius equal to that of the given circle.



Now while MN is given in length and direction, the area of the figure $EFGH$ represents the number of ways the two points can be taken, so that the square will be wholly within the circle.

Let $AE=r$, $MN=2x$, $\angle AEP=\theta$, ϕ=the angle which MN makes with some fixed line, area $EFGH=u$.

Then we have $OA=x$, $AP=\frac{1}{2}x\sqrt{2}$, $\angle EAK=\angle AOP-\angle AEP=\frac{1}{4}\pi-\theta$, $\frac{1}{2}x\sqrt{2}=r\sin\theta$, $EK=r\sin(\frac{1}{4}\pi-\theta)$, area $EOF=r^2(\frac{1}{4}\pi-\theta)-rx\sin(\frac{1}{4}\pi-\theta)=r^2(\frac{1}{4}\pi-\theta-\sin\theta\cos\theta+\sin^2\theta)$, and $u=r^2(\pi-4\theta-4\sin\theta\cos\theta+4\sin^2\theta)$. An element of the circle at the point N is $4xdxd\phi$, or $8r^2\sin\theta\cos\theta d\theta d\phi$. The limits of θ are 0 and $\frac{1}{4}\pi$; and of ϕ, 0 and 2π. Hence the chance in question is

$$\delta=\frac{1}{\pi^2r^4}\int_0^{\frac{1}{4}\pi}\int_0^{2\pi}r^2(\pi-4\theta-4\sin\theta\cos\theta+\sin^2\theta).8r^2\sin\theta\cos\theta d\theta d\phi$$

$$=\frac{16}{\pi}\int_0^{\frac{1}{4}\pi}(\pi-4\theta-4\sin\theta\cos\theta+4\sin^2\theta)\sin\theta\cos\theta d\theta=2-(4/\pi).$$

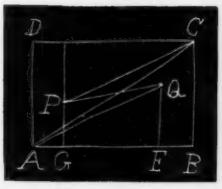
An excellent solution of this problem was also received from G. B. M. ZERR.

136. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

By direct computation find the average distance between two points in the surface of a given rectangle, but on the opposite sides of a diagonal.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the PROPOSER.

Let P, Q be the random points, $AB=a$, $BC=b$, $a^2+b^2=d^2$, $AE=x$, $QE=y$, $CF=u$, $PF=v$. Then $PQ=\sqrt{[(x+u-a)^2+(y+v-b)^2]}$. The limits of x are 0 and a; of u, 0 and a; of y, 0 and bx/a; of v, 0 and bu/a. Let $(x+u-a)=c$, Δ=average distance.



$$\therefore\Delta=\frac{4}{a^2b^2}\int_0^a\int_0^a\int_0^{bx/a}\int_0^{bu/a}\sqrt{[c^2+(y+v-b)^2]}$$

$$\times dxdudydv=\frac{2}{a^2b^2}\int_0^a\int_0^a\int_0^{bx/a}\Big[\Big(\frac{ay+bu-ab}{a^2}\Big)$$

$$\times\sqrt{(a^2c^2+(ay+bu-ab)^2]}-(y-b)\sqrt{[c^2+(y-b)^2]}$$

$$+c^2\log\Big(\frac{ay+bu-ab+\sqrt{[a^2c^2+(ay+bu-ab)^2)}}{ay-ab+a\sqrt{[c^2+(y-b)^2]}}\Big)\Big]dxdudy$$

$$=\frac{2}{3a^5b^2}\int_0^a\int_0^a\{[2a^2c^2-b^2(u-a)^2]\sqrt{[a^2c^2+b^2(u-a)^2]}+[(2a^2c^2-b^2)(x-a)^2]$$

$$\times\sqrt{[a^2c^2+b^2(x-a)^2]}-a^3(2c^2-b^2)\sqrt{[b^2+c^2]}+dc^3(d^2-3a^2)$$

$$+3a^2bc^3\log[c(b+d)]-3a^3bc^2\log a-3a^2bc^2(x-a)\log[b(x-a)$$

$$+\sqrt{[a^2c^2+b^2(x-a)^2]}]-3a^3bc^2\log[\sqrt{(b^2+c^2)}-b]-3a^2bc^2(u-a)\log[b(u-a)$$

$$+\sqrt{[a^2c^2+b^2(u-a)^2]}]\}dxdu$$

$$=\frac{1}{72a^5b^2}\int_0^a\Big[24x[2a^2x^2-3b^3(x-a)^2]\sqrt{[a^2x^2+b^2(x-a)^2]}-16a^3bx^3$$

$$-16a^3b(x-a)^3+(x-a)^4(72b^2d-12a^2d-12d^3-23a^2b)+x^4(12d^3-36a^2d$$

$$+23a^2b)+12a^3x(3b^3-2x^2)\sqrt{[b^2+x^2]}+12a^3(x-a)[2(x-a)^2-3b^2]$$

$$\times\sqrt{[b^2+(x-a)^2]}+60a^2b(x-a)^4\log[(b+d)(x-a)]$$

$$+\frac{24b^4(x-a)^4}{a}\log[(a+d)(x-a)]-96a^2bx^3(x-a)\log\{b(x-a)$$

$$+\sqrt{[a^2x^2+b^2(x-a)^2]}\}-48a^3bx^3\log[\sqrt{(b^2+c^2)}-b]-\frac{24b^4(x-a)^4}{a}$$

$$\times\log\{ax+\sqrt{[a^2x^2+b^2(x-a)^2]}\}+48a^3b(x-a)^3\log\{\sqrt{[b^2+(x-a)^2]}-b\}$$

$$+12a^3b^4\log[x+\sqrt{(b^2+x^2)}]-12a^3b^4\log\{x-a+\sqrt{[b^2+(x-a)^2]}\}$$

$$+36a^2bx^4\log[x(b+d)]-48a^3bx^3\log a+48a^3b(x-a)^3\log a\Big]dx$$

$$=\tfrac{1}{15}\Big[2d\Big(2-\frac{a^2}{b^2}-\frac{b^2}{a^2}\Big)-\frac{a^3+b^3}{d^2}+\frac{2a^3}{b^2}+\frac{2b^3}{a^2}+\Big(\frac{4b^2}{a}-\frac{a^2b^2}{d^3}\Big)\log\Big(\frac{a+d}{b}\Big)$$

$$+\Big(\frac{4a^2}{b}-\frac{a^2b^2}{d^3}\Big)\log\Big(\frac{b+d}{a}\Big)\Big]=\frac{a}{30}[6+(16-\sqrt{2})\log(1+\sqrt{2})],\text{ if } a=b.$$

MISCELLANEOUS.

135. Proposed by LON C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

Find invariants of the second, third, and sixth degrees in the coefficients of a binary quartic.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=u=0$, be the binary quartic.

Then $d^4u/dx^4=24a$, $d^4u/dx^3dy=24b$, $d^4u/dx^2dy^2=24c$, $d^4u/dxdy^3=24d$, $d^4u/dy^4=24e$.

Substitute in u, d^4u/dx^4 for y^4, d^4u/dy^4 for x^4, $-d^4u/dx^3dy$ for xy^3, $-d^4u/dxdy^3$ for x^3y, d^4u/dx^2dy^2 for x^2y^2, and we get $24ae-96bd+144c^2-96bd+24ae=48(ae-4bd+3c^2)$, the invariant of second order.

$d^2u/dx^2=12(ax^2+2bxy+cy^2)$.

$d^2u/dy^2=12(cx^2+2dxy+ey^2)$.

$d^2u/dxdy=12(bx^2+2cxy+dy^2)$.

$\therefore$ The Hessian of this quartic is

$(ax^2+2bxy+cy^2)(cx^2+2dxy+ey^2)-(bx^2+2cxy+dy^2)^2=0$, or $(ac-b^2)x^4+2(ad-bc)x^3y+(ae+2bd-3c^2)x^2y^2+2(be-cd)xy^3+(ce-d^2)y^4=0$.

Write this in the form $Ax^4+4Bx^3y+6Cx^2y^2+4Dxy^3+Ey^4=0$.

Then $24(aE-4bD+6cC-4dB+Ae)=72(ace+2bcd-ad^2-eb^2-c^3)$, is the invariant of the third order.

$\therefore S=ae-4bd+3c^2$, is the invariant of the second order.

$T=ace+2bcd-ad^2-eb^2-c^3$ is the invariant of the third order.

These are the only two ordinary invariants. The invariant of the sixth order being the discriminant which we determine as below.

If we had used the more general form $Ax^4+By^4+Cz^4$, where $x+y+z=0$, then we would have had $a=A+C$, $e=B+C$, $b=C=c=d$.

$\therefore S=BC+CA+AB$, $T=ABC$.

Equating to zero the two differential equations Ax^3-Cz^3, and By^3-Cz^3, we get $Ax^3=By^3=Cz^3$, or $ABCx^3/BC=ABCy^3/AC=ABCz^3/AB$.

$\therefore x^3 : y^3 : z^3=BC : AC : AB$, this in $x+y+z=0$ gives $(BC)^{\frac{1}{3}}+(AC)^{\frac{1}{3}}+(AB)^{\frac{1}{3}}=0$.

$\therefore (BC+AC+AB)^3-27A^2B^2C^2=0$, or $S^3-27T^2=0$.

$\therefore (ae-4bd+3c^2)^3-27(ace+2bcd-ad^2-eb^2-c^3)^2=$the sixth order.

Also solved by *G. W. GREENWOOD*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

169. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

At what rate per cent. must a note be discounted at the end of every quarter of a year in order to produce a discount equivalent to 10% interest for the year?

ALGEBRA.

181. Proposed by J. F. LAWRENCE, Breckenridge, Mo.

Show that $\phi(1)\frac{x}{1+x^2}-\phi(3)\frac{x^3}{1+x^6}+\phi(5)\frac{x^5}{1+x^{10}}-$ *ad inf.* $=\frac{x(1-x^2)}{(1+x^2)^2}$, $\phi(n)$ being the number of integers less than n and prime to it. [From Hall and Knight's *Higher Algebra*, page 358].

182. Proposed by J. F. LAWRENCE, Breckenridge, Mo.

Find the values of $x_1, x_2, x_3, \ldots x_n$ which satisfy the following system of simultaneous equations:

$$\frac{x_1}{a_1-b_1}+\frac{x_2}{a_1-b_2}+\ldots+\frac{x_n}{a_1-b_n}=1.$$

$$\frac{x_1}{a_2-b_1}-\frac{x_2}{a_2-b_2}+\ldots+\frac{x_n}{a_2-b_n}=1.$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\frac{x_1}{a_n-b_1}+\frac{x_2}{a_n-b_2}+\ldots+\frac{x_n}{a_n-b_n}=1.$$

GEOMETRY.

203. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Eng.

Show that two parabolae can always be drawn through the vertices of a triangle to touch its circumcircle at a vertex, and that the axes of these pairs of curves are orthogonal. Show that any triangle may be circumscribed by a conic so that the tangents at each vertex is parallel to the opposite side.

204. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

Construct a triangle, having given an angle, the length of its bisector, and the sum of the including sides. [Phillips and Fisher].

CALCULUS.

168. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The tangent of what Cartesian curve makes an x-intercept always m times as long as the corresponding y-intercept?

169. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Find the value of y from the Eulerian equation

$$y=\int\frac{dx}{(x+\sqrt{3})\sqrt{(x^2+1)}}.$$

MECHANICS.

159. Proposed by J. E. SANDERS, Hackney, Ohio.

Required the time for a tree, considered as a material line of uniform density, length $a=100$ feet, to fall; the tree being inclined $\phi=1'$ from perpendicular.

160. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Given the paracentric acceleration c^2/r^4 and the angular velocity $(n/m)\pi$, to determine the equation of the orbit.

DIOPHANTINE ANALYSIS.

116. Proposed by HARRY S. VANDIVER, Bala, Pa.

If n is an odd positive integer, and 1, n, n', n'', ..., denote all its distinct integral divisors, then $2^n>2(n+1)(n'+1)(n''+1)$.....

117. Proposed by R. W. D. CHRISTIE.

Without the use of the method of continued fractions, solve the equation $x^2-149y^2=1$, and generalize your method. [From the *Educational Times*].

AVERAGE AND PROBABILITY.

145. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

In each quadrant of a given circle, a circle is described at random. A point is taken at random in each of these circles. What is the average area of the quadrilateral formed by joining with straight lines these four points?

146. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

A random straight line crosses a given ellipse; find the chance that two points, taken at random in the ellipse, shall lie on opposite sides of the line.

MISCELLANEOUS.

140. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Prof. Felix Klein asserts that

$$I_1=\int \frac{dz}{\sqrt{[z(1-z)(1-kz)]}}=2\mathrm{sn}^{-1}(\sqrt{z})$$

is a *more* canonical form of the first Legendrian elliptic integral than

$$I_2=\int_0^x \frac{dx}{\sqrt{[(1-x^2)(1-\kappa^2x^2)]}}=\mathrm{sn}^{-1}(x,\kappa).$$

Show this.

141. Proposed by O. W. ANTHONY.

Is f, where f is an operation defined by the equation $f(\mu\nu)=\mu f(\nu)+\nu f(\mu)$, necessarily distributive?

NOTES.

Lewis Neikirk took his Doctor's degree at the University of Pennsylvania on the 17th of June.

Dr. E. R. Hedrick of the Sheffield Scientific School of Yale University, has been elected to the chair of mathematics of the State University of Missouri.

Dr. J. W. Young, who took his Doctor's degree at Cornell this year, has been appointed Assistant Professor of Mathematics at Northwestern University.

Editor Dickson desires to obtain the following back numbers of the MONTHLY to complete his set: Vol. IV, No. 6-7, June-July, 1897; Vol. VI, No. 9-10, October, 1899; Vol. VII, No. 1, January, 1900. Any one having extra copies of one or more of these three numbers will confer a favor by notifying him.

On April 7th, 1903, Editor Finkel was appointed, by the Trustees of the University of Pennsylvania, Special Fellow in Mathematics for the period of one year from the first of September, 1903. In view of this appointment, the Executive Committee of the Board of Trustees of Drury College has granted him a leave of absence for one year. His address, therefore, after September 25th, will be the University of Pennsylvania, Philadelphia, Penn.

In order, therefore, that he may get the greatest good out of the opportunity thus offered for recreation and study, it is desirable that he be relieved as much as possible from the cares and responsibilities of the management of the MONTHLY. It was hoped that a full statement of the arrangement for its management could be made in this issue and the issue was withheld from the mail for over a week awaiting the decision of a gentleman to whom Mr. Finkel's position had been offered. But this gentleman, at the last moment, accepted a permanent position at another place. Hence, at this writing, the matter is not definitely settled. But whatever arrangement is made we wish to assure our readers that the work of the MONTHLY will be carried on, and no pains will be spared on the part of those entrusted to its management to keep up its high standard of efficiency and usefulness.

As during the past year, Dr. Dickson will continue to edit the papers which are offered for publication, and all matters pertaining thereto should be addressed to him at the University of Chicago.

All subscriptions for the coming year should be sent to Mr. W. C. Calland, Treasurer of Drury College, and also all requests for sample copies or missing numbers should be addressed to him.

BOOKS AND PERIODICALS.

The Principles of Mathematics. By Bertrand Russell, A. M., late Fellow of Trinity College, Cambridge. Vol. I., large 8 vo. Cloth. xxix+534 pages. Price $3.50 New York: The Macmillan Co.

The author says that the object of the work is two fold. One of these objects is to prove that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of logical principles. This proof is undertaken in this volume in parts II—VII. In volume II, a demonstration of the same proposition is to be given by strict symbolic reasoning.

The second object of the work, occupying part I, is the explanation of the fundamental concepts which mathematics accepts as indefinable.

The work before us which is addressed in equal measure to philosophers and mathematicians, must not be considered easy reading. Indeed, some chapters, as for example, the chapter on Symbolic Logic, the chapter on Propositional Functions and many others, are quite obstruse. But the chapter on Infinity, the Infinitesimal, and Continuity, and many others are quite valuable to the ordinary student of mathematics.

The second volume, the author informs us, will be addressed exclusively to mathematicians. The whole work will constitute a valuable addition to the library of philosophers and mathematicians alike. B. F. F.

Plane and Solid Geometry, on the Suggestive Method, with Numerous Exercises and a Brief Course on Loci of Equations and on Conic Sections, by A. W. Williamson, Professor of Mathematics in Augustana College, Rock Island, Ill. 8 vo. Cloth. x+283 pp. Price, $1.00.

This work is based on the "heuristic method", and, while small in size, yet it contains more than the usual amount of matter offered in the common geometry course.

Many propositions are combined and easy exercises are given illustrating the important propositions. The book is a good one, and is worthy a cordial reception by teachers of Geometry who are seeking a good book on the subject and presenting it from the standpoint of the "heuristic method." B. F. F.

The American Journal of Mathematics. Edited by Frank Morley and others. Published Quarterly under the Auspices of Johns Hopkins University. Price, $5.00 per year, in advance

No. 3 of Vol. XXV contains the following articles: Isothermal-Conjugate Systems of Lines on Surfaces, by L. P. Eisenhart; Some Differential Equations Connected with Lines on Hypersurfaces, by G. O. James; On the Forms of Sextic Scrolls of Genus Greater than One, by Virgil Snyder; Geometry of the Cuspidal Cubic Cone, by Frederick C. Ferry.

The Annals of Mathematics. Editer by W. E. Byerly and others. Published Quarterly under the Auspices of Harvard University. Price, $2.00 per year, in advance.

No. 4 of Vol. IV, Second Series, contains the following articles: On the Character of Differential Equations, by E. R. Hedrick; On the Uniformity of the Convergence of Certain Absolutely Convergent Series, by Maxime Bocher; The Integral as the Limit of a Sum, and a Theorem of Dahamel's, by W. F. Osgood; On a General Relation of Continued Fractions, by R. E. Moretz; Note on the Equilateral Hyperbola, by J. A. Van Groos; A New Proof of the Generalized Wilson's Theorem, by G. A. Miller; A Sufficient Condition for the Maximum Number of Imaginary Roots of an Equation of the nth Degree, by E. B. Van Vleck.

VOLUME

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. X. AUGUST-SEPTEMBER, 1903. NOS. 8-9.

BIOGRAPHY.

JOHN DANIEL RUNKLE.

BY PROFESSOR H. W. TYLER, MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

John Daniel Runkle was born at Root, N. Y., October 11, 1822, and died at Southwest Harbor, Me., July 8, 1902, near the close of his eighthieth year.

The early years of life on the farm offered little opportunity for study, and he was already twenty-five when he entered the newly established Lawrence Scientific School of Harvard University. His name stands alone in the catalogue of 1848-49 as "student in mathematics." John W. Draper and James E. Oliver were fellow students; Josiah P. Cooke and William T. Harris, resident graduates. He was a member of the first graduating class, of 1851, with Joseph Le Conte and David A. Wells, receiving the degree of Bachelor of Science, and at the same time, for high scholarship, the honorary degree of Master of Arts.

The work of computation for the *Nautical Almanac* was carried on at this time in Cambridge by a staff including, among other men of subsequent eminence, Simon Newcomb, Asaph Hall, George W. Hill, T. H. Safford, and J. M. Van Vleck. Mr. Runkle's connection with the Almanac began in 1849, and continued in some form as late as 1884.

In 1852, he contributed, to the *Astronomical Journal*, papers on the "Elements of Thetis" and on the "Elements of Psyche."

In 1855, his "New tables for determining the values of coefficients, in the perturbative function of planetary motion, which depend upon the ratio of the